New Aspects of Line Bundle Cohomology and Applications to String Phenomenology

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Model building in the $E_8 \times E_8$ heterotic string theory

The effective theory can be specified in terms of 2 pieces of geometrical data:

- a Calabi-Yau threefold $X$
- a slope-zero, polystable, holom. vector bundle $V$ with structure group $G$

**Result:** a four-dimensional model with $N = 1$ supersymmetry, a gauge group given by the commutant of $G$ in $E_8$ and chiral matter.

The simplest class of vector bundles are abelian bundles, i.e. sums of line bundles. Example: $V = \bigoplus_{i=1}^5 L_i$, resulting in an $SU(5)$ GUT.

<table>
<thead>
<tr>
<th>multiplet</th>
<th>bundle</th>
<th>total number</th>
<th>required</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$V$</td>
<td>$\sum_i h^1(X, L_i)$</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>$V^*$</td>
<td>$\sum_i h^1(X, L_i^*)$</td>
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<tr>
<td>5</td>
<td>$\wedge^2 V$</td>
<td>$\sum_{i&lt;j} h^1(X, L_i \otimes L_j)$</td>
<td>$3 + n_H$</td>
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<tr>
<td>5</td>
<td>$\wedge^2 V^*$</td>
<td>$\sum_{i&lt;j} h^1(X, L_i^* \otimes L_j^*)$</td>
<td>$n_H$</td>
</tr>
<tr>
<td>1</td>
<td>$V \otimes V^*$</td>
<td>$\sum_{i,j} h^1(X, L_i \otimes L_j^*)$</td>
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</tbody>
</table>
Line bundle cohomology formulae

topological data of \((X, V)\)

\[ h^\bullet(X, V) \]

global data: cohomology groups

local data
The Euler characteristic

- The Hirzebruch-Riemann-Roch theorem gives

\[ \chi(X, V) = \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, V) = \int_X \text{ch}(V) \cdot \text{td}(X) \]

Main question: is there anything like \( h^i(X, V) = \int_X \text{topological inv}(X, V) \)?
• Nice bundle: all higher cohomologies vanish, then $h^0(X, V) = \chi(X, V)$.

• Example: line bundles on $\mathbb{P}^n$, the Bott formula

\[ h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{k+n}{n} = \frac{1}{n!} (1 + k) \ldots (n + k), \text{ if } k \geq 0, \text{ and } 0 \text{ otherwise.} \]

\[ h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0, \text{ if } 0 < i < n. \]

\[ h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{-k-1}{-n-k-1} = \frac{1}{n!} (-n-k) \ldots (-1-k), \text{ if } k \leq -n-1, \]

and 0 otherwise.

• Notation: $\mathcal{O}_{\mathbb{P}^n}(k)$ is the line bundle whose curvature 2-form is $kJ$, where $J$ is the standard Kähler form on $\mathbb{P}^n$. 
Formulae on complex surfaces
Line bundle cohomology on surfaces: generalities

- We studied: compact toric surfaces, (generalised) del Pezzo surfaces and K3 surfaces.
- General feature: the Picard lattice splits into a number of regions (cones), in each of which the dimension of the zeroth cohomology of $L$ can be computed as the Euler characteristic of $L$ or of some other line bundle $\tilde{L}$.
- Information needed to write down the general formula: the intersection form and the generators of the Mori cone; these give both the regions and the map $L \rightarrow \tilde{L}$ in each region.
- In general, finding the intersection form and the generators of the Mori cone is difficult.
**Complex Surfaces: a toric example**

- Zariski decomposition: \( D = P + N \)
  \[ P = D - N \]
  \( D \) effective, integral; \( P \) nef, rational

- \( h^0(S, \mathcal{O}_S(D)) = h^0(S, \mathcal{O}_S([P])) = h^0(S, \mathcal{O}_S([P])) \)

- vanishing theorem for \( \mathcal{O}_S([P]) \) or \( \mathcal{O}_S([P]) \)

- \( h^0(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S([P])) \)

<table>
<thead>
<tr>
<th>( \Sigma )</th>
<th>( h^0(F_6, \mathcal{O}_{F_6}(D)) )</th>
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<tbody>
<tr>
<td>( \Sigma_{\text{nef}} )</td>
<td>( \chi(F_6, \mathcal{O}_{F_6}(D)) )</td>
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<tr>
<td>( \Sigma_1 )</td>
<td>( \chi(F_6, \mathcal{O}_{F_6}(D - [-\frac{1}{2}D \cdot M_1] M_1)) )</td>
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<td>( \Sigma_2 )</td>
<td>( \chi(F_6, \mathcal{O}_{F_6}(D - (-D \cdot M_2) M_2)) )</td>
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<tr>
<td>( \Sigma_3 )</td>
<td>( \chi(F_6, \mathcal{O}_{F_6}(D - (-D \cdot M_3) M_3)) )</td>
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<tr>
<td>( \Sigma_{1,2} )</td>
<td>( \chi(F_6, \mathcal{O}_{F_6}(D - (-D \cdot (M_1 + M_2)) M_1 - (-D \cdot (M_1 + 2M_2)) M_2)) )</td>
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<tr>
<td>( \Sigma_{1,3} )</td>
<td>( \chi(F_6, \mathcal{O}_{F_6}(D - [-\frac{1}{2}D \cdot M_1] M_1 - (-D \cdot M_3) M_3)) )</td>
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<tr>
<td>$\Sigma$</td>
<td>$h^0(F_6, \mathcal{O}_{F_6}(k_1\mathcal{M}_1 + k_2\mathcal{M}_2 + k_3\mathcal{M}_3))$</td>
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<tr>
<td>$\Sigma_{\text{nef}}$</td>
<td>$1 - k_1^2 + \frac{1}{2}k_2 + k_1k_2 - \frac{1}{2}k_2^2 + \frac{1}{2}k_3 + k_2k_3 - \frac{1}{2}k_3^2$</td>
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<tr>
<td>$\Sigma_1$</td>
<td>$1 + \frac{1}{2}k_2 - \frac{1}{2}k_2^2 + \frac{1}{2}k_3 + k_2k_3 - \frac{1}{2}k_3^2 + k_2 \left\lfloor \frac{1}{2}k_2 \right\rfloor - \left\lfloor \frac{1}{2}k_2 \right\rfloor^2$</td>
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<tr>
<td>$\Sigma_2$</td>
<td>$1 + \frac{1}{2}k_1 - \frac{1}{2}k_1^2 + k_3 + k_1k_3$</td>
</tr>
<tr>
<td>$\Sigma_3$</td>
<td>$1 - k_1^2 + k_2 + k_1k_2$</td>
</tr>
<tr>
<td>$\Sigma_{1,2}$</td>
<td>$1 + \frac{3}{2}k_3 + \frac{1}{2}k_3^2$</td>
</tr>
<tr>
<td>$\Sigma_{1,3}$</td>
<td>$1 + k_2 + k_2 \left\lfloor \frac{1}{2}k_2 \right\rfloor - \left\lfloor \frac{1}{2}k_2 \right\rfloor^2$</td>
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Theorem: line bundle cohomology formula for toric surfaces

Let $S$ be a smooth projective toric surface, and $D$ an effective integral divisor with Zariski decomposition $D = P + N$. Then

$$h^0(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S(\lfloor P \rfloor)).$$

Explicitly, if $D$ lies in the Zariski chamber $\Sigma_{i_1, \ldots, i_n}$, obtained by translating a codimension $n$ face $F$ of the nef cone along the set of dual Mori cone generators $\{M_{i_1}, M_{i_2}, \ldots, M_{i_n}\}$ orthogonal (with respect to the intersection form) to the face $F$, then

$$h^0(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S(D - \sum_{k=1}^{n} [ -D \cdot M_{i_k, \{i_1, \ldots, i_n\}] M_{i_k} )].$$

Similar theorems for generalised del Pezzo surfaces and K3 surfaces, more details in [Brodie, AC, 2009.01275].
Line Bundle Cohomology on CY3
General features

- We studied: CICY three-folds, smooth quotients thereof by freely acting
discrete symmetries, (hypersurfaces) in toric varieties.
- We know empirically that analytic formulae exist for all cohomology
groups. By Serre duality, it is enough to understand the zeroth and the
first cohomologies. So far: the zeroth cohomology (global sections).
- The Picard group splits into various cones, in each of which the zeroth
cohomology can be computed as an index.
- In the Kähler cone $\mathcal{K}(X)$, due to Kodaira’s vanishing theorem

$$h^0(X, L) = \chi(X, L)$$

where the Euler characteristic of $L = \mathcal{O}_X(D)$, on a Calabi-Yau 3-fold is

$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{6} D^3 + \frac{1}{12} c_2(X) \cdot D$$

$$= \frac{1}{6} d_{ijk} k^i k^j k^k + \frac{1}{12} d_{ijk} c_2(X)^{ij} k^k, \quad D = \sum_{i=1}^{h^{1,1}(X)} k^i D_i$$
General features

- We use line bundle cohomology to infer the existence of the flops and to read-off Gromov-Witten invariants.
- Neighbouring $\mathcal{K}(X)$, there are K"ahler cones of flopped manifolds $X'$, which we detect by fitting the zeroth cohomology data to the Euler characteristic in these regions

$$h^0(X, L) = h^0(X', L') = \chi(X', L')$$

- We read off the triple intersection numbers on $X'$ as well as the $c_2(X')$ form. The way in which these differ from the topological data on $X$ is related to Gromov-Witten invariants, which we are able to read off
- In certain cases the effective cone contains other subcones that are not K"ahler cones of flopped manifolds.
Flops

- Flops are co-dimension two surgery operations and isomorphisms in co-dimension one.
- On a three-fold, a flop contracts (cuts out) a number of isolated $\mathbb{P}^1$-curves (rational curves) and replaces them with others.

\[
\begin{array}{ccc}
X & \xrightarrow{\text{flop}} & X' \\
\downarrow \text{contr. 1} & & \downarrow \text{contr. 2} \\
\mathcal{X}_{\text{sing}} & & 
\end{array}
\]

- The flopped manifold $X'$ is Calabi-Yau and has the same Hodge numbers as $X$. The triple intersection numbers and the $c_2$-form change in the following way:

\[
D'^3 = D^3 - \sum_i (D \cdot C_i)^3
\]

\[
c_2(X') \cdot D' = c_2(X) \cdot D + 2 \sum_i D \cdot C_i,
\]

where $C_1, C_2, \ldots, C_N$ are the isolated exceptional $\mathbb{P}^1$ curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ contracted in the flop.
Examples
The manifold 7887, favourable embedding

\[ X = \mathbb{P}^1 \left[ \begin{array}{c} 2 \\ 4 \end{array} \right]^{2,86} \]

\[ L = \mathcal{O}_X(k_1D_1 + k_2D_2) \]

- The positive quadrant: \( \mathcal{K}(X) \).
- Here \( h^0(X, L) = \chi(X, L) \), the Euler characteristic being computed with the following topological data:

\[
\begin{array}{cccccc}
  d_{111} & d_{112} & d_{122} & d_{222} & c_2 \cdot D_1 & c_2 \cdot D_2 \\
  0 & 0 & 4 & 2 & 24 & 44 \\
\end{array}
\]

where \( d_{ijk} = D_i \cdot D_j \cdot D_k \).

- The other cone is \( \mathcal{K}(X') \), and has generators \( \{ \tilde{D}_1' = -D_1' + 4D_2, \tilde{D}_2' = D_2' \} \). The topological data for \( X' \) is given by

\[
\begin{array}{cccccc}
  d_{111}' & d_{112}' & d_{122}' & d_{222}' & c_2' \cdot D_1' & c_2' \cdot D_2' \\
  -64 & 0 & 4 & 2 & 152 & 44 \\
\end{array}
\]

- In the basis \( \{ \tilde{D}_1', \tilde{D}_2' \} \), this data is identical to that for \( X \). Hence \( X' \cong X \).
- The number of isolated curves in the class dual to \( D_1 \) is 64. This is the correct Gromov-Witten invariant.
The manifold 7887, continued

Since \( X' \cong X \), it is not surprising that the zeroth coh. displays a \( \mathbb{Z}_2 \) symmetry

\[
h^0(X, \mathcal{O}_X(k)) = h^0(X, \mathcal{O}_X(Mk)) \quad \text{with} \quad M = \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix}, \quad k = (k_1, k_2)^T
\]

\[
K(X) \quad \chi(X, L)
\]

\[
K(X') \quad \chi(X', L')
\]

<table>
<thead>
<tr>
<th>region in eff. cone</th>
<th>( h^0(X, L = \mathcal{O}_X(k_1D_1 + k_2D_2)) )</th>
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</thead>
<tbody>
<tr>
<td>( k_1 = 0, \ k_2 &gt; 0 )</td>
<td>( \chi(X, L) )</td>
</tr>
<tr>
<td>( k_1 \leq 0, \ k_2 = -4k_1 )</td>
<td>( \chi(X', L') )</td>
</tr>
<tr>
<td>( k_1 \geq 0, \ k_2 = 0 )</td>
<td>( \chi(\mathbb{P}^1, -k_1H_{\mathbb{P}^1}) )</td>
</tr>
<tr>
<td></td>
<td>( \chi(\mathbb{P}^1, k_1H_{\mathbb{P}^1}) )</td>
</tr>
</tbody>
</table>
The manifold 7885

\[ X = \mathbb{P}^1_{\mathbb{P}^4} \left[ \begin{array}{cc} 1 & 1 \\ 4 & 1 \end{array} \right]^{2,86} \]

\[ L = \mathcal{O}_X(k_1 D_1 + k_2 D_2) \]

New feature: the presence of a Zariski chamber.

<table>
<thead>
<tr>
<th>region in eff. cone</th>
<th>( h^0(X, L = \mathcal{O}_X(D = k_1 D_1 + k_2 D_2)) )</th>
<th>( \chi(X, \mathcal{O}_X(D)) )</th>
<th>( \chi(X', \mathcal{O}_{X'}(D')) )</th>
<th>( \chi \left( X', \mathcal{O}_{X'} \left( D' - \left[ \frac{D' \cdot \tilde{C}_2'}{\Gamma'} \right] \Gamma' \right) \right) )</th>
<th>( \chi(\mathbb{P}^1, (D \cdot C_1)H_{\mathbb{P}^1}) )</th>
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<tbody>
<tr>
<td>( \mathcal{K}(X) )</td>
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<tr>
<td>( \overline{\mathcal{K}}(X') \setminus { \mathcal{O}_X } )</td>
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<tr>
<td>( \Sigma )</td>
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<tr>
<td>( k_1 \geq 0, \ k_2 = 0 )</td>
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</tbody>
</table>
The manifold 7863

\[ X = \mathbb{P}^3 \left[ \begin{array}{ccc} 2 & 1 & 1 \\ 2 & 1 & 1 \end{array} \right]^{2,66} \]

\[ L = \mathcal{O}_X(k_1 D_1 + k_2 D_2) \]

New features: infinitely many Kähler cones. The effective cone (in this case the extended Kähler cone) turns out to be irrational.
Heterotic Flop Transitions
Topological transitions in the heterotic string: an open problem

- Topological transitions in the heterotic string context: difficult, due to the presence of internal gauge flux. What happens with the bundle when the manifold undergoes a topological transition?

- We have a proposal for carrying line bundles over flop transitions, understood in terms of the divisor - line bundle correspondence and the observation that flops are co-dimension two surgery operations and isomorphisms in co-dimension one.

- Divisor classes, and hence line bundles, can unambiguously be matched across the transition.

- Things to understand: the anomaly cancellation condition and the changes in the massless spectrum.

- Slope stability of the the bundle (N=1 supersymmetry) may prohibit access to the flopping locus. For enough Kähler moduli, this is not a problem.
Summary and Outlook
An overview of the work to date

- formula on the tetra-quadric CY 3-fold (hypersurface of degree \((2,2,2,2)\) in \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\)) [AC DPhil thesis '13; Buchbinder, AC, Lukas 1311.1941]
- formulae on several Picard number 2, 3 and 4 CICY threefolds and smooth quotients thereof by discrete symmetries [AC, Lukas, 1808.09992]
- (hypersurfaces) in toric varieties [Klaewer, Schlechter, 1809.02547]
- more Picard number 2 CICY 3-folds [Larfors, Schnedeir, 1906.00392]
- formulae on del Pezzo and Hirzebruch surfaces; first mathematical proofs [Brodie, AC, Deen, Lukas, 1906.08769, 1906.08363]
- machine learning implementation [Brodie, AC, Deen, Lukas, 1906.08730]
- certain classes of surfaces understood: theorems for toric surfaces, generalised del Pezzo surfaces and K3 surfaces; simple elliptic fibrations over such surfaces [Brodie, AC, 2009.01275]
- zeroth cohomology on CICY threefolds understood [Brodie, AC, Lukas 2010.06597]
- ...more to come

Case-by-case results computed algorithmically with

- the CICY package [Anderson, Gray, He, Lee, Lukas]
- cohomCalg package [Blumenhagen, Jurke, Rahn, Thorsten, Roschy]
- pyCICY - a Python CICY toolkit [Larfors, Schneider]
There are a number of questions that we would like to address in the near future:

- work out cohomology formulae for manifolds with higher Picard number; automatise / machine learn
- understand the structures present in the higher cohomologies (on threefolds: the first cohomology)
- cohomology formulae for non-abelian bundles: monads, extensions
- implement a bottom-up approach to model building; reinforcement learning of the string landscape
- explore the complex structure dependence of the cohomology formulae; moduli stabilisation