Geometric approach for 3d interfaces at strong coupling

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Outline

• Non-dynamical example: Topological insulator
• The fundamental domain of Maxwell theory and its T-invariant subspace
• Generalization to other duality groups $\Gamma \subset \text{SL}(2, \mathbb{Z})$
• A six-dimensional interpretation
• Other construction of interfaces in four dimensions
• Conclusions and Outlook
Topological insulator

[Kane, Mele ’05,’06], [Bernevig, Hughes, Zhang ’06], [Moore, Balents, ’06], ...

System with **unbroken global U(1) and time-reversal symmetry.**

→ Couple U(1) to **background gauge field** $A$ with local term

$$\frac{\theta}{8\pi^2} F \wedge F$$

**Breaks time-reversal invariance** unless

$$\theta \in \{0, \pi\} \mod 2\pi$$

**trivial phase**  **topological insulator phase**
Constructing an interface

Induces a **Chern-Simons term:**

$$\frac{1}{8\pi} A \wedge F$$

→ **breaks time-reversal invariance**
  (half-integer level)

To restore it, one needs to **add fields on the interface**

- charged 3d Dirac fermion
- topological field theory (gapped phases) e.g. [Seiberg, Witten ‘16]
What if U(1) is dynamical?

\[ \mathcal{L} = \frac{1}{2g^2} F \wedge \ast F + \frac{\theta}{8\pi^2} F \wedge F \]

Has \textbf{SL}(2,\mathbb{Z}) duality [Montonen, Olive '77], [Witten '95]

\[ \tau \rightarrow \frac{a\tau + b}{c\tau + d} , \quad \begin{pmatrix} q_e \\ q_m \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_e \\ q_m \end{pmatrix} \]

with complexified coupling:

\[ \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \]

Time-reversal acts as:

\[ \theta \rightarrow -\theta , \quad \tau \rightarrow -\overline{\tau} \]

anti-holomorphic involution
Time-reversal revisited

Why is \( \theta = \pi \) time-reversal invariant?

\[
\theta = \pi \rightarrow -\pi \sim -\pi + 2\pi = \pi \quad \text{using T generator of SL}(2,\mathbb{Z})
\]

What if one uses S generator instead?

\[
\tau \rightarrow -\frac{1}{\tau} = -\frac{\bar{\tau}}{|\tau|^2}
\]

acts as time-reversal for \(|\tau| = 1\)

Potential for time-reversal invariant phases at strong coupling
Time-reversal in pure Maxwell

F-theory toolkit:
Parametrize the physically inequivalent values for $\tau$ by the complex structure of an auxiliary torus

Weierstrass form:

$$y^2 = x^3 + fx + g$$

$$J(\tau) = \frac{j(\tau)}{1728} = \frac{4f^3}{4f^3 + 27g^2} = \frac{4f^3}{\Delta}$$

time-reversal $\leftrightarrow J \in \mathbb{R}$
Building interfaces

Let $\tau$ vary with respect to one coordinate in flat space $x_3$

For time-reversal invariant system $J(\tau) \in \mathbb{R}$

(can be realized by $f, g \in \mathbb{R}$ $\rightarrow$ realm of real elliptic curves)

Passing through:

- $J = \infty$ topological insulator
  (one expects localized dynamics and indeed $\Delta = 0$)
- $J = 1$ also leads to $\Delta = 0$
  (strongly coupled localized dynamics?)
Other SL(2,Z) subgroups

Assume that the actual duality group is a subgroup of SL(2,Z)

\[ \Gamma_0(N), \Gamma_1(N), \Gamma(N) \subset \text{SL}(2, \mathbb{Z}) \]

in the presence of charged fields

e.g.

\[ \Gamma(N) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) : \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} \]

\[ \rightarrow \text{ Changes fundamental domain or modular curve} \]

We are interested in the time-reversal invariant subspace:

\[ X(\Gamma)_{\mathbb{R}} = \left\{ \tau \in X(\Gamma) : -\bar{\tau} = \tau \right\} = \left\{ \tau \in \mathbb{H} : -\bar{\tau} = \gamma \tau \text{ with } \gamma \in \Gamma \right\} \]
Example:

\[ \Gamma = \Gamma(2) \] generated by

\[ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \]

**Single closed component with three cusps:**

- \( \tau \to i\infty \) topological insulator interface
electric states
- \( \tau = 0 \) related via magnetic states
- \( \tau = 1 \) related via dyonic states

\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ \gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

equivalent of the \( J \)-function, the Hauptmodul, given by elliptic \( \lambda \)-function
Seiberg-Witten theory\footnote{Seiberg, Witten ’94}

$\mathcal{N} = 2$ SYM in 4d with gauge algebra $\mathfrak{su}_2 \rightarrow U(1)$ gauge theory in IR

$$y^2 = (x - u)(x - \Lambda^2)(x + \Lambda^2), \quad u = \frac{1}{2} \text{tr}\phi^2$$

duality group given by: $\Gamma = \Gamma(2)$

- interface crossing $\tau = i\infty$ localized electric fields (W-bosons)
- interface crossing $\tau = 0$ localized magnetic fields (monopole point)
- interface crossing $\tau = 1$ localized dyonic fields (dyon point)

Similar analysis for other $\mathcal{N} = 2$ theories (more exotic AD theories)
Mathematical classification [Snowden ‘11]

Real components of modular curves classified:

Splits into disconnected sets of topological circles with “special points”

- time-reversal invariant configuration sticks to one component
- only certain combinations of interfaces realizable
- localized states from coset representatives (interesting statistics between surfaces)
6d interpretation

Maxwell theory as anti-symmetric 6d tensor $B$ on a torus

$$A = \int_C B, \quad A_D = - \int_C B$$

charged fields form strings coupling to $B$

$\tau$ literally the complex structure of torus

$\Delta = 0 \rightarrow$ 1-cycle pinches $\rightarrow$ massless states

charges by type of 1-cycle $C = q_e C - q_m C$
Congruence subgroups

Demand invariance of certain line operators in 4d

\[ \exp \left( i \int_{L \times (rC + sC)} B \right) = \exp \left( ir \int_{L} A - is \int_{L} A_{D} \right) \]

restricts the identifications: \( \text{SL}(2, \mathbb{Z}) \rightarrow \Gamma \)

(example: \( \Gamma(2) \) if \( r \) and \( s \) are measured mod 2 and the line operators are invariant)

Alternatively: as torsion points on Jacobian

\[ \mathcal{J}(E) = H^{1}(E, \mathbb{R})/H^{1}(E, \mathbb{Z}) \cong \tilde{E} \]

with decomposition:

\[ N\sigma \in H^{1}(E, \mathbb{Z}): B \sim \tilde{A} \land \sigma \]

→ direct relation to congruence subgroups (example: \( \Gamma(2) \) preserves \( \mathbb{Z}_{2} \times \mathbb{Z}_{2} \) torsion)
Other geometrical interfaces

So far: varied the shape of a torus along one direction (+ subtleties involving time-reversal)

Why stop there?

Vary instead: genus, fluxes, ...

Some control over chiral degrees of freedom via anomaly

Again use interplay between real geometry and time-reversal to secure protection of localized states
Conclusions

• Use duality to explore time-reversal invariant regions in moduli space

• Uncovers connection to real elliptic curves for Maxwell theory

• Deduce localized degrees of freedom from time-reversal invariance of interfaces (associated to special points in moduli space, e.g. cusps)

• Passes tests where we have control ($\mathcal{N} = 2$ supersymmetric theories)

• Embedding into higher-dimensional theories makes the geometric perspective more apparent

• Explore further (more exotic) possibilities for geometrically engineered interfaces
Outlook

• Anomalies in duality groups [Tachikawa, Yonekura ’17], [Cordova, Freed, Lam, Seiberg ’19], [Hsieh, Tachikawa, Yonekura ’19]

• Non-Abelian groups with 1-form center symmetries [Aharony, Seiberg, Tachikawa ’13],… (relation to MW-torsion, e.g. [Hajouji, Oehlmann ’19])

• Realization within higher-dimensional internal space (Spin-7 manifolds) e.g. recent progress in [Cvetic, Heckman, Rochais, Torres, Zoccarato ’20]

• Tests beyond SUSY