On stratification diagrams, algorithmic spectrum estimates and vector-like pairs in F-theory

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With M. Cvetič, R. Donagi, L. Lin, M. Liu, F. Rühle – 2020.06***
Obtain (MS)SM from String theory construction . . .

- $E_8 \times E_8$ [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Anderson Gray He Lukas '10], . . .

- Type II [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00].
  [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01],

- F-theory [Krause Mayrhofer Weigand '12], [Cvetič Klevers Pena Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], . . .

. . . including vector-like spectra

- Why vector-like spectra? Higgs fields matter & are characteristic feature of QFTs
  - $E_8 \times E_8$: [Bouchard Donagi '05], [Braun He Ovrut Pantev '05], [Anderson Gray Lukas Palti '10 & '11], . . .
  - F-theory: [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]
In this talk

- Recent progress to understand *vector-like spectra* in F-theory
- Based on
  - Machine learning (c.f. L. Lin at *String pheno 2020*)
  - Analytic insights (Brill Noether theory, stratifications . . .)
- Today: Focus on analytics
Gauge degrees **localized** on 7-branes $S \subset B_3$

Zero modes **localized** on matter curves $C_R \subset S$

$G_4$-flux and matter surface $S_R$ define line bundle $\mathcal{L}_R$ on $C_R$

Vector-like pairs:

- massless chiral modes $\leftrightarrow h^0(C_R, \mathcal{L}_R)$
- massless anti-chiral modes $\leftrightarrow h^1(C_R, \mathcal{L}_R)$

Typically, $h^i(C_R, \mathcal{L}_R)$ hard to determine:

- By definition – non-topological data
- Oftentimes, $\mathcal{L}_R$ not pullback from $B_3$

Coherent sheaves on $B_3 \leftrightarrow$ Freyd categories [S. Posur '17], [M.B., S. Posur '19]

Deformation $C_R \rightarrow C'_R$ can lead to jumps

$$h^i(C_R, \mathcal{L}_R) = (h^0, h^1) \rightarrow h^i(C'_R, \mathcal{L}'_R) = (h^0 + a, h^1 + a)$$
## Strategy

### Geometric setup

- Realistic F-theory geometries computationally too involved
- ⇒ Learn from simpler geometries first
- Choice of geometry:
  
  \[
  \text{Curve} \leftrightarrow C(c) = V(P(c)) \text{ hypersurface in } dP_3
  \]
  
  \[
  \text{Line bundle} \leftrightarrow \mathcal{L}(c) = \mathcal{O}_{dP_3}(D_L)|_{C(c)}
  \]

### Challenge

Find \( h^0(C(c), \mathcal{L}(c)) \equiv h^0(c) \) as function of the complex structure \( c \)
How to find $h^0(C(c), L) \equiv h^0(c)$?

1. Pullback line bundle admits Koszul resolution:

$$0 \to \mathcal{O}_{dP_3}(D_L - D_C) \xrightarrow{P(c)} \mathcal{O}_{dP_3}(D_L) \to L \to 0$$

2. Obtain long exact sequence in sheaf cohomology:

$$
\begin{array}{ccccccc}
0 & \to & H^0(D_L - D_C) & \to & H^0(D_L) & \to & H^0(L) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & H^1(D_L - D_C) & \to & H^1(D_L) & \to & H^1(L) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & H^2(D_L - D_C) & \to & H^2(D_L) & \to & 0 \\
& & \downarrow & & & & 0 \\
& & 0 & & & & 0
\end{array}
$$

3. Sometimes: $0 \to H^0(L) \to H^1(D_L - D_C) \xrightarrow{M_\varphi(c)} H^1(D_L) \to H^1(L) \to 0$

4. By exactness: $h^0(L) = \ker(M_\varphi(c))$

$\Rightarrow$ Study $\ker(M_\varphi(c))$ as function of complex structure $c$
Motivation
Analysis of jumps
Summary and Outlook

h^0\text{-stratifications}
Jumps from Brill-Noether theory
Jumps from curve splittings

Example: \( g = 3, \chi = 1 \ (d = 3) \)

- \( C(c) = V(P(c)) \) and \( P(c) = c_1x_1^3x_2^3x_3^2x_4 + \cdots + c_{12}x_3^2x_4^3x_5^3x_6^3 \)
- For \( D_L = H + 2E_1 - 2E_2 - E_3 \) find

\[
0 \to H^0(L) \to \mathbb{C}^3 \xrightarrow{M_\varphi(c)} \mathbb{C}^2 \to H^1(L) \to 0, \quad M_\varphi = \begin{pmatrix} c_3 & c_2 & c_1 \\ 0 & c_{12} & c_{11} \end{pmatrix}
\]

- \( h^0(L) = 3 - \text{rk}(M_\varphi(c)) \) & stratification of curve geometries:

<table>
<thead>
<tr>
<th>\text{rk}(M_\varphi)</th>
<th>\text{explicit condition}</th>
<th>\text{curve splitting}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>((c_3c_{11}, c_3c_{12}, c_2c_{11} - c_1c_{12}) \neq 0)</td>
<td>(C^1)</td>
</tr>
<tr>
<td>1</td>
<td>(c_3 = 0, c_2c_{11} - c_1c_{12} = 0)</td>
<td>(C^2)</td>
</tr>
<tr>
<td>1</td>
<td>(c_1 = c_2 = c_3 = 0)</td>
<td>(B_2 \cup \mathbb{P}^1_b)</td>
</tr>
<tr>
<td>1</td>
<td>(c_{11} = c_{12} = 0)</td>
<td>(\mathbb{P}^1_a \cup B_1)</td>
</tr>
<tr>
<td>0</td>
<td>(c_1 = c_2 = c_3 = c_{11} = c_{12} = 0)</td>
<td>(\mathbb{P}^1_a \cup A \cup \mathbb{P}^1_b)</td>
</tr>
</tbody>
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**Motivation**

**Analysis of jumps**

**Summary and Outlook**

- **$h^0$-stratifications**
  - Jumps from Brill-Noether theory
  - Jumps from curve splittings

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**Stratification diagram**

$h^0 = 1$

$h^0 = 2$

$h^0 = 3$

- **Brill-Noether theory:** $C^2$ smooth, irreducible but line bundle divisor special
- **Curve splittings:** Factoring off $\mathbb{P}_a^1, \mathbb{P}_b^1$ leads to jump
Example 2: $g = 5$, $\chi = 0$ ($d = 4$)

- $P(c) = c_1 x_1^3 x_2^4 x_3^2 x_4^2 + \cdots + c_{16} x_3^3 x_4^4 x_6^3$
- $D_L = H + E_1 - 4 E_2 + E_3$
- Koszul resolution gives

\[
h^0(\mathcal{L}) = 7 - \text{rk}(M_{\varphi}(c))
\]

- $M_{\varphi} = \begin{pmatrix}
    c_{15} & c_{11} & c_7 & 0 & 0 & 0 & 0 \\
    0 & c_{10} & c_6 & c_3 & c_{11} & c_7 & 0 \\
    c_{12} & c_6 & c_3 & 0 & c_7 & 0 & 0 \\
    0 & c_5 & c_2 & 0 & c_6 & c_3 & c_7 \\
    c_8 & c_2 & 0 & 0 & c_3 & 0 & 0 \\
    0 & c_{14} & c_{11} & c_7 & 0 & 0 & 0 \\
    0 & c_1 & 0 & 0 & c_2 & 0 & c_3
\end{pmatrix}$

\[\Rightarrow\] Study $\text{rk}(M_{\varphi}(c))$ as function of $c$
Brill-Noether theory [1874 Brill, Noether] – more modern exposition in [Mumford '75], [Griffiths, Harris '94] ...
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Example on torus $C_1 \cong \mathbb{C}/\Lambda = \text{Jac}(C_1)$

\[ (-1) \cdot q \]

\[ p \]

$\mathcal{O}_{C_1}(p - q)) = 0 \quad \rightarrow \quad h^0(\mathcal{O}_{C_1}(0)) = 1$

\[ G^0_0 = \{ \mathcal{L} , \ d = n = 0 \} \]

\[ \cong \{ q \in \mathbb{C}/\Lambda , \ q \neq 0 \} \]
Example on torus $C_1 \cong \mathbb{C}/\Lambda = \text{Jac}(C_1)$

$\begin{align*}
\text{h}^0(\mathcal{O}_{C_1}(p - q)) &= 0 \\
\text{h}^0(\mathcal{O}_{C_1}(0)) &= 1
\end{align*}$

$G_0^0 = \{\mathcal{L}, \ d = n = 0\}$
$\cong \{q \in \mathbb{C}/\Lambda, \ q \neq 0\}$

$G_0^1 = \{\mathcal{L}, \ d = 0, \ n = 1\}$
$\cong \{q = 0 \in \mathbb{C}/\Lambda\}$
**Example on torus** $C_1 \cong \mathbb{C}/\Lambda = \text{Jac}(C_1)$

\[
\begin{align*}
\text{Abel-Jacobi map gives } & \varphi_d : \text{Div}_d(C) \to \text{Jac}(C) \cong \mathbb{C}^g / \Lambda \\
G_d^n & = \{ \varphi_d(\mathcal{L}), \ h^0(C, \mathcal{L}) = n \} \subseteq \text{Jac}(C) \\
\dim G_d^n & \geq \rho(d, n, g) = g - n \cdot (n + \chi) \\
\dim G_d^n & = \rho \text{ for generic curves} \quad [1980 \text{ Griffiths, Harris}]
\end{align*}
\]
Brill-Noether theory \[1874\] Brill, Noether – more modern exposition in [Mumford '75], [Griffiths, Harris '94] ...

Example on torus \( C_1 \cong \mathbb{C}/\Lambda = \text{Jac}(C_1) \)

\[
\begin{align*}
    (-1) \cdot q & \quad \text{p} \\
    h^0 (\mathcal{O}_{C_1}(p - q)) = 0 & \quad \rightarrow \quad h^0 (\mathcal{O}_{C_1}(0)) = 1 \\
    G_0^0 = \{ \mathcal{L} , d = n = 0 \} & \quad \cong \{ q \in \mathbb{C}/\Lambda , q \neq 0 \} \\
    G_0^1 = \{ \mathcal{L} , d = 0, n = 1 \} & \quad \cong \{ q = 0 \in \mathbb{C}/\Lambda \}
\end{align*}
\]

General picture

- Abel-Jacobi map gives \( \varphi_d : \text{Div}_d(C) \rightarrow \text{Jac}(C) \cong \mathbb{C}^g / \Lambda \)
- \( G_d^n = \{ \varphi_d(\mathcal{L}) , h^0(C, \mathcal{L}) = n \} \subseteq \text{Jac}(C) \)
- \( \dim G_d^n \geq \rho(d, n, g) = g - n \cdot (n + \chi) \)
- \( \dim G_d^n = \rho \) for generic curves \[1980\] Griffiths, Harris

<table>
<thead>
<tr>
<th>( h^0 )</th>
<th>( h^1 )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-3</td>
</tr>
</tbody>
</table>

Example on torus $C_1 \cong \mathbb{C}/\Lambda = \text{Jac}(C_1)$

$\psi_d : \text{Div}_d(C) \to \text{Jac}(C) \cong \mathbb{C}^g/\Lambda$

$G^0_d = \{ \mathcal{L} \mid d = n = 0 \}
\cong \{ q \in \mathbb{C}/\Lambda \mid q \neq 0 \}$

$G^1_d = \{ \mathcal{L} \mid d = 0, n = 1 \}
\cong \{ q = 0 \in \mathbb{C}/\Lambda \}$

General picture

- Abel-Jacobi map gives $\varphi_d : \text{Div}_d(C) \to \text{Jac}(C) \cong \mathbb{C}^g/\Lambda$
- $G^n_d = \{ \varphi_d(\mathcal{L}) \mid h^0(C, \mathcal{L}) = n \} \subseteq \text{Jac}(C)$
- $\dim G^n_d \geq \rho(d, n, g) = g - n \cdot (n + \chi)$
- $\dim G^n_d = \rho$ for generic curves [1980 Griffiths, Harris]

$\Rightarrow$ Upper bound for $h^0$ on generic curves [Watari, 16]
Gluing *local* sections

\[ h^0 = 1 \]

\[ h^0 = 2 \]

\[ h^0 = 3 \]
Gluing local sections

$h^0 = 1$

$h^0 = 2$

$h^0 = 3$
Gluing local sections

\[ h^0 = 1 \]

\[ h^0 = 2 \]

\[ h^0 = 3 \]

\[ \deg \left( L|_{P^1_a} \right) = -2 \]
\[ g(P^1_a) = 0 \]
\[ h^0 \left( L|_{P^1_a} \right) = 0 \]

\[ \deg \left( L|_{B_2} \right) = 5 \]
\[ g(B_2) = 2 \]
\[ h^0 \left( L|_{B_2} \right) = 4 \]
Gluing \textit{local} sections II

\begin{align*}
h^0 &= 1 \\
h^0 &= 2 \\
h^0 &= 3
\end{align*}
Gluing *local* sections II

\[ h^0 = 1 \]

\[ h^0 = 2 \]

\[ h^0 = 3 \]

\[ C^1 \]

\[ C^2 \]

\[ B_1 \cup \mathbb{P}^1_b \]

\[ \mathbb{P}^1_a \cup B_2 \]

\[ \mathbb{P}^1_a \cup A \cup \mathbb{P}^1_b \]
Gluing local sections II

$h^0 = 1$

$h^0 = 2$

$h^0 = 3$

$\deg \left( \mathcal{L}_{|\mathbb{P}^1_b} \right) = -2$

$g(\mathbb{P}^1_b) = 0$

$h^0 \left( \mathcal{L}_{|\mathbb{P}^1_b} \right) = 0$

$\deg (\mathcal{L}|_A) = 7$

$g(A) = 0$

$h^0 (\mathcal{L}|_A) = 8$

$\deg \left( \mathcal{L}_{|\mathbb{P}^1_a} \right) = -2$

$g(\mathbb{P}^1_a) = 0$

$h^0 \left( \mathcal{L}_{|\mathbb{P}^1_a} \right) = 0$
Quality assessment of counting procedure

- **Quick**: Uses only topological data (genus, chiral index)
- **But**: Relative position of bundle divisor and intersections of curve components matters [Cayley 1889, Bacharach 1886]
  - Systematically **over**estimates # of independent conditions
  - Obtain **under**estimate # of global sections
- Application to our data base:
  - 83 pairs \((D_C, D_L)\) with complex structure deformations: \(\sim 1.8 \times 10^6\) data sets
  - Counting procedure can be applied to \(\sim 38\%\)
  - Accuracy \(\sim 98.5\%\)

- **Lead-offs**:
  1. Sufficient conditions for jump
  2. Algorithmic \(h^0\)-spectrum estimate
Algorithmic estimate for $h^0$-spectrum

Max. degenerate curves

https://github.com/homalg-project/SheafCohomologyOnToricVarieties

Caveat: Check that $\tilde{C}$ is irreducible
Algorithmic estimate for $h^0$-spectrum

$V(x_1) \cup \tilde{C} \quad \cdots \quad V(x_6) \cup \tilde{C}$

Max. degenerate curves

https://github.com/homalg-project/SheafCohomologyOnToricVarieties
Algorithmic estimate for $h^0$-spectrum

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$V(x_1) \cup \tilde{C}$
$V(x_1^2) \cup \tilde{C}$

$V(x_1) \cup \tilde{C}$
$V(x_1) \cup V(x_6) \cup \tilde{C}$

$V(x_5) \cup V(x_6) \cup \tilde{C}$

$V(x_6) \cup \tilde{C}$
$V(x_6^2) \cup \tilde{C}$

$\max$ degenerate curves

https://github.com/homalg-project/SheafCohomologyOnToricVarieties

Estimate $h^0$-spectrum from lower bounds at subset of nodes

Implemented in package $H0Approximator$

Caveat: Check that $\tilde{C}$ is irreducible
Algorithmic estimate for $h^0$-spectrum
Algorithmic estimate for $h^0$-spectrum

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\[ V(x_1) \cup \tilde{C} \]
\[ \cdots \]
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Max. degenerate curves

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Algorithmic estimate for $h^0$-spectrum

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Max. degenerate curves

Caveat: Check that $\tilde{C}$ is irreducible

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Max. degenerate curves

Caveat: Check that $\tilde{C}$ is irreducible

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Max. degenerate curves

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Max. degenerate curves

\[ V(x_1) \cup \tilde{C} \]
\[ V(x_1^2) \cup \tilde{C} \]
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...
Algorithmic estimate for \( h^0 \)-spectrum

- Estimate \( h^0 \)-spectrum from lower bounds at subset of nodes
- Implemented in package \( H0\text{Approximator} \)

\[ \text{https://github.com/homalg-project/SheafCohomologyOnToricVarieties} \]
Algorithmic estimate for $h^0$-spectrum

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- Estimate $h^0$-spectrum from lower bounds at subset of nodes
- Implemented in package H0Approximator
- Caveat: Check that $\tilde{C}$ is irreducible
Summary

- Computing vector-like spectra in global F-theory models is hard
- We study how vector-like spectrum changes over moduli space of curve (↔ qualitatively different from previous bundle cohomology studies)
- Insights from interplay between
  - Machine learning techniques (decision trees)
  - Analytic insights (Brill-Noether theory, stratification diagrams)
- Finding in $dP_3$: Factor off (rigid) $\mathbb{P}^1$s ↔ jumps
- Results:
  1. Formulate sufficient condition for jump
  2. Implement quick (mostly based on topological data) $h^0$-spectrum approximator

$H^0\text{Approximator}$: https://github.com/homalg-project/SheafCohomologyOnToricVarieties/
Outlook

- **Technical extensions:**
  - non-pullback bundle and “fractional” bundles
  - stratification for several curves in one global F-theory model

- **Conceptual:**
  - Vector-like spectra for pseudo-real representations
  - Non-vertical $G_4$ (flux moduli dependence!)
  - (Geometric) symmetries protecting vector-like pairs

- **Practical:**
  - model building
  - (S)CFTs
  - swampland program
Thank you for your attention!