Probing Higgs Bundles for Local $G_2$-Manifolds

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Seminar Series on String Phenomenology

Overview

1. Introduction and Motivation
2. Local $G_2$-Manifolds and 7d twisted SYM
3. Higgs Bundles and a Colored Supersymmetric QM
4. Abelian Higgs Backgrounds
5. Summary and Outlook
Introduction and Motivation

M-theory on local (i.e. non-compact) $G_2$-manifolds engineers minimally supersymmetric gauge theories in 4d.

[Acharya, 2000], [Witten, 2001], [Atiyah, Witten, 2003], [Pantev, Wijnholt, 2009],
[Braun, Cizel, H, Schafer-Nameki, 2018], [Barbosa, Cvetič, Heckman, Lawrie, Torres, Zoccarato, 2019],
[Cvetič, Heckman, Rochais, Torres, Zoccarato, 2020], [H, 2020], [Acharya, Kinsella, Svanes, 2020],
[Acharya, Najjar, Foscolo, Svanes, 2020]

The 4d gauge theories describe the localized degrees of freedom in compact $G_2$-manifolds.

[Acharya, 1998], [Halverson, Morrison, 2015], [Guio, Jockers, Klemm, Yeh, 2017],
[Braun, Schafer-Nameki, 2017], [Braun, Del Zotto, 2017], [Braun, 2019], [Xu, 2020]
F-theory methods relying on Higgs bundles and their spectral covers can be applied to study the physics of local $G_2$-manifolds. [Beasley, Heckman, Vafa, 2009], [Hayashi, Kawano, Tatar, Watari, 2009], [Marsano, Saulina, Schafer-Nameki, 2009], [Marsano, Saulina, Schafer-Nameki, 2010], [Blumenhagen, Grimm, Jurke, Weigand, 2010], [Donagi, Wijnholt 2011], [Donagi, Wijnholt 2014]

Supersymmetric sigma models probing the geometries give insight into non-perturbative effects and indices.

[Alvarez-Gaumé, Witten, 1981], [Witten, 1982], [Pantev, Wijnholt, 2009],

[Braun, Cizel, H, Schafer-Nameki, 2018], [H, 2020]

Question: What determines the 4d physics engineered by local $G_2$-manifolds in M-theory?
ALE Fibered, Local $G_2$-Manifolds

Geometric data

Local $G_2$-Manifold: \( \mathbb{C}^2/\Gamma_{\text{ADE}} \hookrightarrow X_7 \rightarrow M_3 \)

Fibral 2-Spheres: \( \sigma_I \in H_2(\mathbb{C}^2/\Gamma_{\text{ADE}}, \mathbb{R}) \)

Hyperkähler Triplet: \( (\omega_1, \omega_2, \omega_3) \in H^2(\mathbb{C}^2/\Gamma_{\text{ADE}}, \mathbb{R}) \)

The Higgs field collects the Kähler periods

Higgs field: \( \phi_I = \left( \int_{\sigma_I} \omega_i \right) dx^i \in \Omega^1(M_3) \)

where \( I = 1, \ldots, \text{rank} \ g_{\text{ADE}} \).
Singularities and Supersymmetric 3-cycles

Singularity Enhancement at $x \in M_3$ : $\phi_I(x) = 0$ (isolated)

The vanishing cycles trace out 3-spheres:
Questions

What is the global structure of the network of supersymmetric 3-spheres and the M2-brane instantons?

What do the M2-branes wrapped on the supersymmetric 3-spheres descend to in an effective 7d field theory description?

How can their contribution to the 4d superpotential be computed from the 7d field theory description?
Previous Work

M-theory on $\mathbb{R}^{1,3} \times X_7$

Twisted 7d SYM on $\mathbb{R}^{1,3} \times M_3$

4d $\mathcal{N} = 1$ Field theory

- [Acharya, Witten, 2001]
- [BCHLTZ, 2019]
- [CHRTZ, 2020]
- [Barbosa, 2019]
- [Harvey, Moore, 1999]
- [H, 2020]
- [Pantev, Wijnholt, 2009]
- [Braun, Cizel, H, Schafer-Nameki, 2018]
Effective 7d Physics

M-theory on the local $G_2$-manifold $X_7$ gives a

Partially twisted 7d SYM on $\mathbb{R}^{1,3} \times M_3$
with gauge group $G_{ADE}$.

Complex bosonic 1-form on $M_3$:

\[ \varphi = \phi + iA \in \Omega^1(M_3, g_{ADE}) \]

Supersymmetric vacua are solutions to the Hitchin system:

\[ i(F_A)_{ij} + [\phi_i, \phi_j] = 0, \quad (d_A \phi)_{ij} = 0, \quad * d_A * \phi = 0 \]
Zero modes along $M_3$ are determined by

$$H = \frac{1}{2} \left\{ Q, Q^\dagger \right\}, \quad Q = d + [\varphi, \cdot]$$

and counted by the cohomologies

$$H^*_Q(M_3, g_{ADE}).$$

Alternatively, consider perturbative zero modes

$$\chi_a \in \Omega^*(M_3, g_{ADE}) \leftrightarrow \text{Codimension 7 Singularity}$$

The zero modes are recovered from

**Mass Matrix** : \[ M_{ab} = \int_{M_3} \langle \chi_b, Q \chi_a \rangle \]

**Yukawa Couplings** : \[ Y_{abc} = \int_{M_3} \langle \chi_c, [\chi_a, \chi_b] \rangle \]
Colored $\mathcal{N} = 2$ SUSY Quantum Mechanics

Bosonic Coordinates on $M_3$ : $x^i$, $i = 1, 2, 3$

Fermions in $x^*(TM_3)$ : $\psi^i$, $i = 1, 2, 3$

Color fermions in $x^*(\text{ad}G_{\text{ADE}})$ : $\lambda^\alpha$, $\alpha = 1, \ldots, \dim g_{\text{ADE}}$

Lagrangian : 
\[
L = \frac{1}{2} \dot{x}^i \dot{x}_i - \frac{1}{2} \dot{\phi}_i \phi_{\lambda,i} + i \bar{\psi}^i \nabla_\tau \psi_i + i \bar{\lambda}^\alpha D_\tau \lambda_\alpha \\
- (D_{(i} \phi_{j)}) \lambda \bar{\psi}^i \psi^j + i (F_{ij}) \lambda \bar{\psi}^i \psi^j + \zeta (\bar{\lambda}^\alpha \lambda_\alpha - 1)
\]

Hilbertspace : $\mathcal{H}_{\text{phys.}} = \Lambda(M_3, g_{\text{ADE}})$

Supercharge : $Q = d + [\varphi, \cdot]$
Bosonic Coordinates on $M_3$:

$x^i, \quad i = 1, 2, 3$

Fermions in $x^* (TM_3)$:

$\psi^i, \quad i = 1, 2, 3$

Color fermions in $x^* (\text{ad} G_{\text{ADE}})$:

$\lambda^\alpha, \quad \alpha = 1, \ldots, \text{dim } g_{\text{ADE}}$
Supercharge:
\[ Q = \bar{\psi}_i (\dot{x}^i - \phi^i_\lambda) \]

Perturbative Groundstates:
\[(x, \lambda) \in M_3 \times \mathfrak{g}_{\text{ADE}} \text{ with } \phi_\lambda(x) = 0\]

1/2-BPS instantons are piecewise solutions to the flow equations

Flow line instanton:
\[ \dot{x}^i - \phi^i_\lambda = \dot{x}^i - ic^{\alpha}_{\beta \gamma} \phi^i_{\alpha} \bar{\lambda}^\beta \lambda^\gamma = D_\tau \lambda^\alpha = 0. \]
Abelian Higgs Backgrounds (split)

Abelian solutions to the BPS equations: \( d_A = d, \phi = \phi_I H^I \).

The 1-form Cartan components \( \phi_I \) are harmonic up to sources

\[
d\phi_I = 0, \quad *d* \phi_I = \rho_I \quad \rightarrow \quad \phi_I = df_I, \quad \Delta f_I = \rho_I,
\]

which are supported on codimension \( \geq 2 \) subloci in \( M_3 \).

Root \( \alpha \in g_{ADE} \) \( \leftrightarrow \) Witten SQM into \( M_3 \) with supercharge \( Q^{(\alpha)} = d + \alpha^I df_I \)
Yukawa Couplings

\[ Y_{abc} = \int_{M_3} \langle \chi_c^{(\gamma)} , [\chi_a^{(\alpha)} , \chi_b^{(\beta)}] \rangle = \int_{M_3} d^3x_0 \int D\psi D\bar{\psi} \exp \left[ i \left( S^{(\alpha)}[x_a, \psi_a, \bar{\psi}_a] + S^{(\beta)}[x_b, \psi_b, \bar{\psi}_b] + S^{(\gamma)}[x_c, \psi_c, \bar{\psi}_c] \right) \right] \]

\[ = \sum_{\Gamma_{abc}} (\pm) \Gamma_{abc} \exp \left( -\int_{\Gamma_a} \alpha^l \phi_l - \int_{\Gamma_b} \beta^l \phi_l + \int_{\Gamma_c} \gamma^l \phi_l \right) \]

\[ = \sum_{\Gamma_{abc}} (\pm) \Gamma_{abc} \exp \left[ -\text{Vol} (S_3^{\Gamma_{abc}}) \right] \rightarrow \text{[Harvey, Moore, 1999]} \]

\[ \chi_r^{(\alpha)} = \lim_{T \to -\infty (1+i\delta)} \frac{e^{-iHT} \psi_r e^{iHT}}{e^{-iE_0,T} \langle \chi_r | \psi_r^{(\alpha)} \rangle} \equiv \psi_r^{(\alpha)} \bigg|_{-\infty} , \quad 0 < \delta \ll 1 , \]

\[ D\psi D\bar{\psi} = \prod_{-\infty < \tau < 0} d \{ x, \psi, \bar{\psi} \}_{a,\tau} d \{ x, \psi, \bar{\psi} \}_{b,\tau} d \{ x, \psi, \bar{\psi} \}_{c,\tau} \]

\[ -\infty < \tau < 0 \quad x_a, -\infty = x_a \quad x_b, -\infty = x_b \quad 0 < \tau < \infty \quad x_c, \infty = x_c \]
Effective 4d Theory (split)

We find:

1. Mathematics fixing effective 4d physics is Fukaya’s Morse theory with multiple Morse functions, [Fukaya, 1997]

2. Chiral and conjugate chiral multiplets in 4d counted by

\[ H^1(M_3, \partial(\alpha) M_3) , \ H^2(M_3, \partial(\alpha) M_3) \]

respectively. [Pantev, Wijnholt, 2009], [Braun, Cizel, H, Schafer-Nameki, 2018]

3. Yukawa couplings given by a cup-product on \( H^*(M_3, \partial(\alpha) M_3) \)
Abelian Higgs Backgrounds (non-split)

Abelian solutions to the BPS equations: \( d_A = d, \quad \phi = \text{diag}(\Lambda_K) \).

With eigenvalue 1-forms \( \Lambda \) harmonic up to source terms

\[
d \Lambda_K = \ast j_K, \quad \ast d \ast \Lambda_K = \rho_K,
\]

which are supported on codimension \( \geq 1 \) subloci in \( M_3 \). (TCS)

Two classes of solutions distinguished by their spectral cover:

\[
\mathcal{C} = \{(x, \Lambda_K(x)) \mid x \in M_3\} \subset T^*M_3 \rightarrow \begin{cases} 
\Lambda_K \text{ globally defined} \\
\Lambda_K \text{ connected by branch cuts}
\end{cases}
\]
Colored SQM for Non-split Spectral Covers

The eigenvalue 1-forms of the Higgs field are interchanged along paths linking the branch locus

\[ \text{Monodromy Action : } \phi \rightarrow g\phi g^{-1} \]
\[ \text{Color Mixing : } E^\alpha \rightarrow gE^\alpha g^{-1} \]

Monodromy orbit \([\alpha]\) \[\leftrightarrow\]

Witten SQM into \(C_k\) with supercharge \(Q([\alpha]) = d + \phi[\alpha]\)

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Probing Higgs Bundles for Local G\(_2\)-Manifolds
Effective 4d Theory (non-split)

We find:

1. Gauge symmetry determined by stabilizer of $\phi$
2. Chiral and conjugate chiral multiplets in 4d counted by

$$H_{\text{Nov.}}^1(C_k, \phi[\alpha]), \quad H_{\text{Nov.}}^2(C_k, \phi[\alpha])$$

3. Yukawa couplings given by a cup-product on $H_{\text{Nov.}}^*(C_k, \phi[\alpha])$
Summary

We introduced a colored SQM probing Higgs bundles of local $G_2$-manifolds, and establish the correspondence

\[ \text{Euclidean M2 Instantons} \leftrightarrow \text{Instantons of Colored SQM} \]

For abelian backgrounds we compute instanton effects giving the

Mass Matrix : $M_{ab} = \int_{M_3} \langle \chi_b, Q \chi_a \rangle$

Yukawa Couplings : $Y_{abc} = \int_{M_3} \langle \chi_c, [\chi_a, \chi_b] \rangle$.

and determine the effective 4d physics.
Outlook

- Extend on non-split Higgs fields and Novikov cohomologies
- Analyze fluxed T-brane solutions
  [Barbosa, Cvetič, Heckman, Lawrie, Torres, Zoccarato, 2019]
- Understand the M-theory origin of the colored SQM
  [Harvey, Moore, 1999]
- Explore other Higgs bundle vacua
  [Cvetič, Heckman, Rochais, Torres, Zoccarato, 2020]
The End
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Higgs Bundles and a Colored Supersymmetric QM

Abelian Higgs Backgrounds

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