VANISHING ORDERS AND U(1) CHARGES IN F-THEORY

Nikhil Raghuram (Virginia Tech) · 28 July 2020

Based on upcoming work with w/ Andrew Turner
GENERAL MOTIVATIONS

BROAD QUESTIONS
Which massless spectra can occur in F-theory vacua?
▶ What types of gauge groups?
▶ Which representations of light charged matter?
How are these spectra realized in F-theory?

Important for a few reasons:

PHYSICS
▶ A good way to explore the F-theory landscape and swampland
   ▶ Which consistent supergravity spectra can be realized in F-theory?
   ▶ Could lead to new constraints, string constructions ...

MATHEMATICS
▶ Many physical properties of F-theory models are encoded in geometry
▶ Can teach us about mathematical properties of elliptic fibrations, Calabi–Yau manifolds, etc.
Many interesting aspects of this to explore for U(1)’s.

- What U(1) charges can massless matter have in F-theory?
- In 6D, an infinite swampland of charge spectra [Taylor, Turner ’18]
  - Infinite families of models with massless matter having arbitrarily large charges that satisfy anomaly cancellation conditions
  - Not known which of these models can be realized in F-theory

It’s worth better understanding massless U(1) charges in F-theory

- In particular, what are the geometric features of F-theory models realizing different charges?

**TODAY’S QUESTIONS**

Can we make general statements about how different charges are realized in F-theory?
Can U(1) charges be determined prior to resolution?
OVERVIEW OF F-THEORY

Describe a model using an elliptically-fibered CY manifold

Nonabelian Gauge Algebras
Divisors in base with singular fibers
- Singularity types $\leftrightarrow$ gauge algebra
- Charged matter at codim-two loci where singularity type enhances
- Enhanced singularity type determines representation

U(1) Algebras
Extra rational sections of fibration
- Charged matter still at codim-two loci with enhanced singularity types
- U(1) charge determined by behavior of section
1. After resolving singularities, elliptic fibers at certain loci may take shape of affine ADE diagrams

**CODIMENSION ONE**

\[ \hat{A}_4 \ (SU(5) \text{ Gauge Algebra}) \]

**CODIMENSION TWO**

\[ \hat{A}_5 \ (SU(5) \text{ Fundamental Matter}) \]

Determine gauge algebra, matter representations from resolution

- At codimension-one, wrapping M2 branes on components gives roots of gauge algebra (in dual M-theory picture)
  - For non-simply-laced algebras, monodromy identifies components
- Wrapping M2 branes on extra components at codimension-two gives weights of charged matter
NONABELIAN GAUGE ALGEBRAS

2. For a model in Weierstrass form

\[ y^2 = x^3 + f x z^4 + g z^6 \quad \Delta = 4f^3 + 27g^2 \]

the Kodaira table relates singularity types to vanishing orders of \( f, g, \Delta \)

<table>
<thead>
<tr>
<th>Singularity Type</th>
<th>Algebra</th>
<th>( \text{ord}(f) )</th>
<th>( \text{ord}(g) )</th>
<th>( \text{ord}(\Delta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_1 )</td>
<td>( - )</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( I_n )</td>
<td>( A_{n-1} )</td>
<td>( \text{su}(n) ) or ( \text{sp}(\lfloor \frac{n}{2} \rfloor) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( II )</td>
<td>( - )</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( III )</td>
<td>( A_1 )</td>
<td>( \text{su}(2) )</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( IV )</td>
<td>( A_2 )</td>
<td>( \text{su}(3) ) or ( \text{su}(2) )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( I_0^* )</td>
<td>( D_4 )</td>
<td>( \text{so}(8) ) or ( \text{so}(7) ) or ( g_2 )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( I_n^* )</td>
<td>( D_{n+4} )</td>
<td>( \text{so}(2n+8) ) or ( \text{so}(2n+7) )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( IV^* )</td>
<td>( E_6 )</td>
<td>( e_6 ) or ( f_4 )</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( III^* )</td>
<td>( E_7 )</td>
<td>( e_7 )</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>( II^* )</td>
<td>( E_8 )</td>
<td>( e_8 )</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Nonabelian gauge algebra can often be read off from codimension-one orders of vanishing of \( f, g, \Delta \).

- There are also simple rules to test for monodromy.
NONABELIAN GAUGE ALGEBRAS

For nonabelian charged matter, representation can be found using Katz–Vafa method [Katz, Vafa ’96]

▶ Determine gauge algebras associated with codimension-one singularities (G) and codimension-two singularities (H)

▶ Break adjoint of \( H \) into reps of \( G \).

▶ Charged matter rep can be read off branching pattern

You can often determine \( G, H \) from the orders of vanishing of \( f, g, \Delta \)

▶ Strictly speaking, Kodaira classification only holds in codimension-one

▶ But you can often get away with using it at codimension-two

You can often determine nonabelian matter representations without resolution, at least heuristically.
U(1)'s in f-theory

U(1)'s come from extra rational sections of the elliptic fibration
- We'll always assume there's at least one section, the zero section
- There may be more than one section (even an infinite number)
- Sections form a finitely-generated group under elliptic curve addition:

  Mordell–Weil Group: \( \mathbb{Z}^r \oplus G \)
  - \( G \) is the finite torsion subgroup, which is unimportant for today
  - \( r \) is the Mordell–Weil Rank

The resulting abelian gauge algebra is \( U(1)^r \)
- Roughly, each U(1) is associated with a generating section
U(1) CHARGED MATTER

Matter still occurs at codim-two loci with enhanced singularity type

Wrapping M2 branes on extra components still gives matter

But the U(1) charge is determined by the behavior of the section

\[ q = \sigma(\hat{s}) \cdot c \]

\( \hat{s} \) The generating section
\( \sigma(\hat{s}) \) The Shioda map, a homomorphism from MW group to the Neron-Severi group
\( c \) An extra component of the fiber

The charge can be determined by examining the resolved geometry

CODIMENSION ONE

\( SU(5) \times U(1) \) Gauge Algebra

CODIMENSION TWO

\( 5_{1/5} \) Matter

\( 5_{-4/5} \) Matter
ABELIAN WEIERSTRASS MODELS

Suppose our elliptic fibration is global Weierstrass form

\[ y^2 = x^3 + f x z^4 + g z^6 \]

\[[x : y : z] \equiv [\lambda^2 x : \lambda^3 y : \lambda z]\]

Suppose we want a model with a U(1) gauge algebra (MW group \( \mathbb{Z} \))

- Need a generating section \( \hat{s} \) that generates \( \mathbb{Z} \)
- \( \hat{s} \) described by components \([\hat{x} : \hat{y} : \hat{z}]\) solving Weierstrass equations
- \( \hat{x}, \hat{y}, \hat{z} \) are holomorphic sections of line bundles on base
  - In principle, they can be rational
  - Rescale to remove denominators, remove shared factors if possible
- It’s also convenient to define

\[ \hat{w} = 3\hat{x}^2 + f\hat{z}^4 \]

Q: Is there something like the Katz–Vafa method for U(1) charges?
Not using only orders of vanishing of \( f, g, \Delta \)

- Example: Singlets occur at codim-two loci where \((f, g, \Delta)\) vanish to orders \((0, 0, 2)\), regardless of charge
But what if we look at section components?
Previously observed that, for charged singlets, the orders of vanishing of the section components are correlated with charge

- The algebraic structure of the F-theory models is closely linked to these orders of vanishing [NR’17]

<table>
<thead>
<tr>
<th>Charge</th>
<th>$\hat{z}$</th>
<th>$\hat{x}$</th>
<th>$\hat{y}$</th>
<th>$\hat{w}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$q = 1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$q = 2$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$q = 3$</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>$q = 4$</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
</tr>
</tbody>
</table>

Is this part of a more general pattern?

- If so, maybe we could use this pattern to read off U(1) charges?

**CHALLENGE** Tough to construct F-theory models with larger charges
To guess how sections admitting larger charges behave:

1. Start with a U(1) model with only charge \( q = 1 \) matter:

\[
y^2 - f_9^2 = (x - f_6) \left( x^2 + f_6 x + \hat{f}_{12} - f_6^2 \right)
\]

\[\hat{s} = [f_6 : f_9 : 1]\]

- At \( \hat{f}_{12} = f_9 = 0 \), there's an \( I_2 \) fiber with an extra component \( c \)
- The generating section \( \hat{s} \) satisfies

\[\sigma(\hat{s}) \cdot c = 1,\]

where \( \sigma \) is the Shioda map

2. There are also sections \( n \hat{s} \) generated from \( \hat{s} \) by elliptic curve addition for any integer \( n \)

3. Since \( \sigma \) is a homomorphism, we also have

\[\sigma(n \hat{s}) \cdot c = n \sigma(\hat{s}) \cdot c = n\]

4. The section \( n \hat{s} \) behaves in a way that looks like it admits charge \( n \)
SINGLETS

For the $n$ sections, look at the orders of vanishing at $\hat{f}_{12} = f_9 = 0$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{z}$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>$\hat{x}$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>32</td>
</tr>
<tr>
<td>$\hat{y}$</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>19</td>
<td>27</td>
<td>37</td>
<td>48</td>
</tr>
<tr>
<td>$\hat{w}$</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>64</td>
</tr>
</tbody>
</table>

Agree with previous models

PATTERN

The components of the $n$ sections vanish to orders

$$\text{ord}(\hat{z}) = \frac{1}{2} \left( \frac{n^2}{2} - \frac{(n \mod 2)}{2} \right)$$

$$\text{ord}(\hat{x}, \hat{y}, \hat{w}) = (2, 3, 4) \times \text{ord}(\hat{z}) + (0, 1, 1) \times (n \mod 2)$$

Components for a generating section should vanish to the same orders at a genuine $q = n$ locus.

[NR '17]
SINGLETS

PATTERN

\[
\text{ord}(\hat{z}) = \frac{1}{2} \left( \frac{q^2}{2} - \frac{(q \mod 2)}{2} \right)
\]

\[
\text{ord}((\hat{x}, \hat{y}, \hat{w}) = (2, 3, 4) \times \text{ord}(\hat{z}) + (0, 1, 1) \times (q \mod 2)
\]

Why this pattern?

▶ Similar numbers appear for valuations of elliptic divisibility sequences corresponding to \(p\)-adic elliptic curves [Stange ’11]

Do similar patterns hold in other situations?

▶ What if matter is charged under both a \(U(1)\) and a nonabelian gauge algebra?

IDEA Use higher sections in a model with \(g \oplus u(1)\) gauge algebra
For a Weierstrass model with a $\mathfrak{g} \oplus \mathfrak{u}(1)$ algebra:

$$q = \sigma(\hat{s}) \cdot c = S \cdot c + \sum_{I,J} (S \cdot \alpha_I)C^{-1}_{IJ}(T_J \cdot c)$$

- $S$ Homology class of section
- $c$ Extra curve in codim-2 fiber supporting a weight of matter
- $\alpha_I$ Curve in singular fibers supporting simple root of $\mathfrak{g}$
- $T_J$ Divisor found by fibering $\alpha_J$ over codim-1 locus
- $C^{-1}$ Inverse Cartan matrix for $\mathfrak{g}$
U(1) CHARGES & NONABELIAN MATTER

\[ q = \sigma(\hat{s}) \cdot c = S \cdot c + \sum_{I,J} (S \cdot \alpha_I) C^{-1}_{IJ} (T_J \cdot c) \]

Matter charged under $g$ can have fractional $u(1)$ charges

▷ Due to the inverse Cartan matrix
▷ Non-trivial contribution when section hits one of the $\alpha_I$ at codimension-one
▷ Singlets still have integer charges

Allowed fractional charges controlled by $S \cdot \alpha_I$

▷ Which $\alpha_I$ is hit by the section?
▷ Codimension-one phenomenon
▷ Related to global structure of gauge group [Cvetic, Lin '17]
**U(1) CHARGES & NONABELIAN MATTER**

**IDEA**

Use higher sections in a model with $\mathfrak{g} \oplus u(1)$ gauge algebra

- As before, if generating section satisfies
  \[ \sigma(\hat{s}) \cdot c = q, \]

  at a codimension-two locus, the $n\hat{s}$ section satisfies
  \[ \sigma(n\hat{s}) \cdot c = nq, \]

- Find orders of vanishing for the $n\hat{s}$ sections at this locus
- In models with genuine charge $nq$ matter, generating section should vanish to same orders

We’ve done this exercise for simply-laced $\mathfrak{g}$ with generic reps:

- $\mathfrak{su}(n)$ Fundamental and Antisymmetric
- $\mathfrak{so}(n)$ Vector, Spinors up through $\mathfrak{so}(14)$
  - $\mathfrak{e}_6$ 27 Representation
  - $\mathfrak{e}_7$ 56 Representation

In total, around 550 sets of orders of vanishing... and there’s a pattern
PROPOSAL

Consider a model with a $\mathfrak{g} \oplus u(1)$ algebra, where $\mathfrak{g}$ is simply-laced

- Codim-one locus with singularity type $G$ (the universal cover of $\mathfrak{g}$)
  - For singlets, take $G$ to be “SU(1)”

- Matter at a codim-two locus where singularity type enhances to $H$

**Codimension-One Orders of Vanishing**

$$\text{ord}_{(1)}(\hat{z}) = 0 \quad (\text{ord}_{(1)}(\hat{x}), \text{ord}_{(1)}(\hat{y}), \text{ord}_{(1)}(\hat{w})) = \vec{\tau}_G(\mathcal{I})$$

**Codimension-Two Orders of Vanishing**

$$\text{ord}_{(2)}(\hat{z}) = \frac{1}{2} \left( \frac{d_G}{d_H} q^2 + \left( C_G^{-1} \right)_{\mathcal{I}\mathcal{I}} - \left( C_H^{-1} \right)_{\mathcal{J}\mathcal{J}} \right)$$

$$(\text{ord}_{(2)}(\hat{x}), \text{ord}_{(2)}(\hat{y}), \text{ord}_{(2)}(\hat{w})) = (2, 3, 4) \times \text{ord}_{(2)}(\hat{z}) + \vec{\tau}_H(\mathcal{J})$$

**$\mathcal{I}$ (or $\mathcal{J}$)** An integer ranging from 0 to rank($G$) (or rank($H$))

- Roughly, component of resolved fiber hit by section

**$\vec{\tau}_G(\mathcal{I})$** Triplet of integers (given by particular expressions)

**$d_G$** Number of elements in the center of $G$

**$(C_G^{-1})_{\mathcal{I}\mathcal{I}}$** $\mathcal{I}'$th diagonal entry of inverse Cartan matrix (or 0 if $\mathcal{I} = 0$)
PROPOSAL, CONT.

Codimension-One Orders of Vanishing

\[ \text{ord}_{(1)}(\hat{z}) = 0 \quad (\text{ord}_{(1)}(\hat{x}), \text{ord}_{(1)}(\hat{y}), \text{ord}_{(1)}(\hat{w})) = \vec{\tau}_G(\mathcal{I}) \]

Codimension-Two Orders of Vanishing

\[ \text{ord}_{(2)}(\hat{z}) = \frac{1}{2} \left( \frac{d_G}{d_H} q^2 + \left( c_{SU(N)}^{-1} \right)_{\mathcal{I}\mathcal{I}} - \left( c_H^{-1} \right)_{\mathcal{J}\mathcal{J}} \right) \]

\[ (\text{ord}_{(2)}(\hat{x}), \text{ord}_{(2)}(\hat{y}), \text{ord}_{(2)}(\hat{w})) = (2, 3, 4) \times \text{ord}_{(2)}(\hat{z}) + \vec{\tau}_H(\mathcal{J}) \]

For \( G = SU(N) \), \( \mathcal{I} \) ranges from 0 to \( N - 1 \):

\[ \vec{\tau}_{SU(N)}(\mathcal{I}) = (0, u_N(\mathcal{I}), u_N(\mathcal{I})) \quad u_N(\mathcal{I}) = \min(\mathcal{I}, N - \mathcal{I}) \]

\[ \left( c_{SU(N)}^{-1} \right)_{\mathcal{I}\mathcal{I}} = \frac{\mathcal{I}(N - \mathcal{I})}{N} \quad d_{SU(N)} = N \]

For SU(N) fundamentals, H is SU(N+1)
The formulas seem to work in a variety of models in the literature:

- U(1) model with $q = 1, 2$ matter in [Morrison, Park ’12]
- Toric hypersurface models in [Klevers, Mayorga-Pena, Oehlmann, Piragua, Reuter ’14]
- U(1) model with $q = 1, 2, 3, 4$ matter in [NR ’17]
- SU(5), SO(10), $E_6$ and $E_7$ models in [Küntzler, Schäfer-Nameki ’14]
- The SU(5) × U(1)$^2$ and SU(4) × U(1)$^2$ models from [Borchmann, Mayrhofer, Palti, Weigand ’13]

Whenever we’ve used these formulas to derive the charge spectrum of a model, the results have agreed with anomaly cancellation.

**POTENTIAL BENEFITS**

- Gives a way to read off charges (at least up to sign) without resolving
- This is a formula for general charges
  - Could be used for exploring the F-theory landscape/swampland
SU(5) EXAMPLE

Consider a Weierstrass model \( y^2 = x^3 + fxz^4 + gz^6 \) with

\[
\begin{align*}
  f &= -\frac{1}{48} \left( b_{1,0}^2 + 4\sigma c_{2,1} \right)^2 + \frac{1}{2} \sigma^2 b_{0,0} b_{1,0} c_{1,2} + \sigma^3 \left( c_{1,2} c_{3,1} - \sigma b_{0,0}^2 c_{0,4} \right) \\
  g &= -\frac{1}{1728} \left( b_{1,0}^2 + 4\sigma c_{2,1} \right)^3 - \frac{1}{12} f \left( b_{1,0}^2 + 4\sigma c_{2,1} \right) + \frac{1}{4} \sigma^4 b_{0,0}^2 c_{1,2}^2 \\
  &\quad - \sigma^5 c_{0,4} \left( b_{0,0}^2 c_{2,1} - b_{1,0} b_{0,0} c_{3,1} - \sigma c_{3,1}^2 \right)
\end{align*}
\]

This model has an SU(5) on \( \{ \sigma = 0 \} \) and a U(1) with generating section

\[
\begin{align*}
  \hat{x} &= \frac{1}{12} b_{0,0}^2 \left( b_{1,0}^2 - 8\sigma c_{2,1} \right) + \sigma b_{1,0} b_{0,0} c_{3,1} + \sigma^2 c_{3,1}^2 \\
  \hat{y} &= \frac{1}{2} \sigma \left[ b_{0,0}^2 b_{1,0} \left( b_{0,0} c_{2,1} - b_{1,0} c_{3,1} \right) \\
  &\quad - \sigma b_{0,0} \left( b_{0,0}^3 c_{1,2} - 2b_{0,0} b_{2,1} c_{2,1} c_{3,1} + 3b_{1,0} c_{3,1}^2 \right) - 2\sigma^2 c_{3,1}^3 \right] \\
  \hat{z} &= b_{0,0}
\end{align*}
\]

There is \( 5_{6/5} \) matter at \( \{ \sigma = b_{0,0} = 0 \} \) with SU(5)\( \rightarrow \)SU(6) enhancement

\[
\begin{align*}
  \text{ord}_{\sigma=0} (\hat{x}, \hat{y}, \hat{z}, \hat{w}) &= (0, 1, 0, 1) \\
  \text{ord}_{\sigma=b_{0,0}=0} (\hat{x}, \hat{y}, \hat{z}, \hat{w}) &= (2, 3, 1, 4)
\end{align*}
\]
SU(5) EXAMPLE, CONTINUED

\[
\text{ord}_{\sigma=0}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) = (0, 1, 0, 1) \quad \text{ord}_{\sigma, b_0, 0=0} = (2, 3, 1, 4)
\]

\[
\tilde{\tau}_{SU(N)}(I) = (0, 1, 1) \times \min(I, N - I)
\]

\[
(C_{SU(N)}^{-1})_{\mathcal{II}} = \frac{I(N - I)}{N}
\]

AT CODIMENSION ONE The singularity type is SU(5)

\[
\text{ord}_{\sigma=0}(\hat{x}, \hat{y}, \hat{w}) = \tilde{\tau}_{SU(5)}(I) = (0, 1, 1) \times \min(I, 5 - I)
\]

\[
= (0, 1, 1)
\]

Therefore, \(I\) is 1 or 4, and \((C_{SU(5)}^{-1})_{\mathcal{II}} = \frac{4}{5}\)
SU(5) EXAMPLE, CONTINUED

\[ \text{ord}_{\sigma=0}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) = (0, 1, 0, 1) \quad \text{ord}_{\sigma, b_0, 0=0} = (2, 3, 1, 4) \]

\[ \bar{\tau}_{SU(N)}(I) = (0, 1, 1) \times \min(I, N-I) \quad \left( C_{SU(N)}^{-1} \right)_{II} = \frac{I(N-I)}{N} \]

**AT CODIMENSION TWO** The singularity type enhances to SU(6).

\[ \text{ord}_{(2)}(\hat{x}, \hat{y}, \hat{w}) = (2, 3, 4) \times \text{ord}_{(2)}(\hat{z}) = \bar{\tau}_{SU(6)}(J) = (0, 0, 0) \]

Therefore, \( J \) is 0, and we take \( (C_{SU(6)}^{-1})JJ \) to be 0.

Since \( d_{SU(N)} = N \) and \( (C_{SU(5)}^{-1})_{II} = \frac{4}{5} \), we have

\[ \text{ord}_{(2)}(\hat{z}) = \frac{1}{2} \left( \frac{d_{SU(5)}}{d_{SU(6)}}q^2 + \left( C_{SU(5)}^{-1} \right)_{II} - \left( C_{SU(6)}^{-1} \right)_{JJ} \right) \]

\[ = \frac{5}{12}q^2 + \frac{4}{10} = 1 \]

Therefore, \( |q| = \frac{6}{5} \), as expected!
Formulas seem to work for other simply-laced gauge algebras and representations.

Extends naturally if there are multiple $U(1)$’s.

Slight generalization of formulas seems to work for bifundamentals.

Similar types of numbers/expressions appear for $p$-adic valuations of elliptic divisibility sequences [Stange ’11].
CONCLUSIONS

In summary, information about U(1) charges seems to be encoded in orders of vanishing of the section components.

FUTURE DIRECTIONS

▶ More rigorous understanding/confirmation of these patterns, either mathematically or physically
▶ Extension to non-simply laced algebras
▶ More exotic matter reps, such as symmetric rep of SU(N)
▶ Superconformal matter
▶ Are similar formulas possible for other descriptions of elliptic fibers (e.g. Tate form, cubic in \(\mathbb{P}^2\), ...)

Thank you!